

ON DEGENERATE SIGMA-FUNCTIONS IN GENUS TWO

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ABSTRACT. We obtain explicit expressions for genus 2 degenerate sigma-function in terms of genus 1 sigma-function and elementary functions as solutions of a system of linear PDEs satisfied by the sigma-function. By way of application we derive a solution for a class of generalized Jacobi inversion problems on elliptic curves, a family of Schrödinger-type operators on a line with common spectrum consisting of a point and two segments, explicit construction of a field of three-periodic meromorphic functions. Generators of rank 3 lattice in \mathbb{C}^2 are given explicitly.

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INTRODUCTION

The concept of a sigma-function in higher genus was introduced by F. Klein [1] in 1886 as an extensive generalization of elliptic Weierstrass sigma-function [2]. The importance of a sigma-function lies in the fact that it is a convenient generator of Abelian functions in g complex variables, i.e. meromorphic multiply periodic functions that possess the maximal number $2g$ of periods. From this viewpoint sigma-function in genus 2 was studied since the time of Klein, and the classical results were well documented in Baker monograph [3].

Theory of sigma-functions progresses in several ways. The first approach considers a generalization called in [4] Kleinian sigma-function $\sigma(u) = \exp(\frac{1}{2}u^t\eta\omega^{-1}u)\theta(\omega^{-1}u; \omega, \omega')$, which is a modular invariant representative of the class of theta functions¹ with $u \in \mathbb{C}^g$. Dealing with this expression for the multivariate sigma-function the authors have examined fields of Abelian functions associated with hyperelliptic curves, described Jacobi and Kummer varieties as algebraic varieties, developed contemporary applications of Abelian functions to completely integrable equations of theoretical and mathematical physics. Further theoretical developments in study of sigma and its relation to theta and tau functions are achieved in [5, 6]. Modular definition of sigma-function for generic algebraic curves is proposed in [7] whilst the case of (n, s) -curves and, in particular, of hyperelliptic curves is studied in this context in [8].

Another approach, aimed to construct series expansions for multivariate sigma-functions, is developed in sequential papers [9–12]. This is a powerful technique for obtaining sigma-series explicitly as well as an elegant theory derived from unfolding Pham singularities. The theory gives many byproducts. For example, the canonical basis of second kind differentials associated to the first kind differentials are constructed without introducing Kleinian bi-differential unlike the first approach; in their turn the second kind differentials can be used for constructing the bi-differential. Algebraic identities and addition laws in the field of Abelian functions

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¹Here Riemann θ -function is used. As usual, ω and η denote first and second kind integrals along \mathfrak{a} -cycles, and ω' denotes first kind integrals along \mathfrak{b} -cycles.

are easily obtained from a sigma-series expansion, as well as Hirota bilinear equations for integrable systems associated with a curve. The theory is applicable for both hyper- and non-hyperelliptic plane curves.

The equivariant approach based on covariance with respect to transformations of an algebraic curve is proposed in [13–15]. For hyperelliptic curves the transformations are induced by $\mathrm{SL}(2, \mathbb{C})$ action. Identities for multivariate Abelian functions arise as finite-dimensional irreducible representations of the corresponding algebra $\mathfrak{sl}(2, \mathbb{C})$; and the covariant form of Kleinian bi-differential is constructed. It was shown that the Hirota derivative plays the role of a partial intertwining operator in the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ and so can serve for generation of algebraic invariants [16]. A generalization of bilinear Hirota operators possessing an equivariance property is applied for constructing a basis in the space of Abelian functions with poles of at most a given order [17].

There are also computational approach where series expansions for multivariate sigma-functions in higher genera and algebraic identities between the corresponding Abelian functions are constructed by numerical methods independently of the second approach, but partly inspired by it. Series for some sigma-functions in genera 3, 4, 6 were obtained [18–22] as well as identities for Abelian functions of the kind $\wp_{[k]}(u) = -\partial^{[k]} \log \sigma(u)$.

The present paper is essentially based on the theory of constructing series for multivariate sigma-functions, mostly on the paper [11]. The theory originates from Weierstrass's definition [2] of sigma-function as the entire function depending on the three variables $(u; g_2, g_3) \in \mathbb{C} \times \mathbb{C}^2$ and satisfying the set of differential equations

$$\begin{aligned} (1) \quad & Q_0(\sigma) = 0, \quad Q_2(\sigma) = 0, \\ & Q_0(\sigma) = -u\sigma_u + 4g_2\sigma_{g_2} + 6g_3\sigma_{g_3} + \sigma, \\ & Q_2(\sigma) = -\frac{1}{2}\sigma_{u,u} - \frac{1}{24}g_2u^2\sigma + 6g_3\sigma_{g_2} + \frac{1}{3}g_2^2\sigma_{g_3} \end{aligned}$$

with initial condition $\sigma(u; 0; 0) = u$; here g_2 and g_3 are parameters of Weierstrass elliptic curve $y^2 = 4x^3 - g_2x - g_3$. A multivariate sigma-function in genus g is an entire function $\sigma(u; \lambda)$ of $3g - m$ complex variables $(u; \lambda) \in \mathbb{C}^g \times \mathbb{C}^{2g-m}$, where m is modality (in the hyperelliptic case, we are dealing with in the present paper, $m = 0$, for more details see [12]). The multivariate sigma-function is defined by a set of heat equations [11] similar to (1) with initial conditions in the form of so called Schur-Weierstrass polynomials [9].

In this paper we study cases when sigma-function in genus 2 can be represented as an aggregate of sigma-function in genus 1 and elementary functions. For $g = 2$ sigma-function depends on six variables $(u; \lambda) \in \mathbb{C}^2 \times \mathbb{C}^4$, where λ is the set of parameters of a genus 2 curve

$$(2) \quad y^2 = x^5 + \lambda_4x^3 + \lambda_6x^2 + \lambda_8x + \lambda_{10}.$$

In fact, curve (2) has *actual genus* 2 only if a certain constraint is imposed on λ . We say that the curve has *virtual genus* 2 and focus on the cases when its actual genus is lower. That is the cases when genus 2 sigma-function can be expressed in terms of genus 1 sigma-function and elementary functions. Strata of the space of parameters λ corresponding to fixed actual genera of (2) are analyzed in Sections 2 and 3. Then we proceed to our main question by carrying out an analysis of the system of linear partial differential equations, so called heat equations in a non-holonomic frame [11], that are satisfied by sigma-function in Section 4, and derive our main result in Section 5.

In Section 6 we apply the result to a few selected problems: solution of a generalized Jacobi inversion problem, construction of a Schrödinger type operator with spectrum composed of two segments and a point, description of the structure of a field of three periodic functions in two complex variables. In the course of our research we also obtain a stratification of the space of parameters of genus 2 curves (2) with respect to the rank of a period lattice corresponding to the curve.

1. NOTATION

Below in the paper we consider the space \mathcal{C} of genus 2 curves of the form

$$(3) \quad x^5 - y^2 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10} = 0.$$

The parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ run over \mathbb{C}^4 . Genus 2 sigma-function is denoted by $\sigma(u; \lambda)$, where $u = (u_3, u_1)$. We assign Satō weights to the variables by the rule $\deg \lambda_i = i$, $\deg u_k = -k$. Accordingly, with $\deg x = 2$ and $\deg y = 5$ the left hand side of (3) is homogeneous of weight 10. It is important, that most of functions and operators appearing below respect Satō weights, in particular, $\deg \sigma(u; \lambda) = -3$.

In what follows we also deal with a family of genus 1 curves

$$(4) \quad X^3 - Y^2 + \gamma_4 X + \gamma_6 = 0,$$

here $\deg X = 2$ and $\deg Y = 3$. To avoid confusion we denote the corresponding genus 1 sigma-function by $\sigma(u_1)$, which stands for standard Weierstrass sigma-function with invariants $(g_2, g_3) = (-4\gamma_4, -4\gamma_6)$.

For brevity we use the notation ∂_x in the place of $\partial/\partial x$.

2. STRATIFICATION OF THE SPACE OF PARAMETERS

The space of parameters Λ is naturally stratified into three strata: Λ_2 , Λ_1 , and Λ_0 which correspond to curves in genus $g = 2$, 1, and 0 respectively.

Proposition 1. *The space Λ is a disjoint union $\Lambda_2 \cup \Lambda_1 \cup \Lambda_0$ and*

$$\begin{aligned} \Lambda_2 &= \{\lambda \in \mathbb{C}^4 \mid \Delta(\lambda) \neq 0\}, \\ \Lambda_1 &= \{\lambda \in \mathbb{C}^4 \mid \Delta(\lambda) = 0, \Gamma(\lambda) \neq 0\}, \\ \Lambda_0 &= \{\lambda \in \mathbb{C}^4 \mid \Gamma(\lambda) = 0\}, \end{aligned}$$

where

$$(5) \quad \begin{aligned} \Delta(\lambda) &= 3125\lambda_{10}^4 - 3750\lambda_{10}^3\lambda_6\lambda_4 + 2000\lambda_{10}^2\lambda_8^2\lambda_4 + 2250\lambda_{10}^2\lambda_8\lambda_6^2 \\ &\quad - 1600\lambda_{10}\lambda_8^3\lambda_6 + 256\lambda_8^5 - 900\lambda_{10}^2\lambda_8\lambda_4^3 + 825\lambda_{10}^2\lambda_6^2\lambda_4^2 + 560\lambda_{10}\lambda_8^2\lambda_6\lambda_4^2 \\ &\quad - 630\lambda_{10}\lambda_8\lambda_6^3\lambda_4 + 108\lambda_{10}\lambda_6^5 - 128\lambda_8^4\lambda_4^2 + 144\lambda_8^3\lambda_6^2\lambda_4 - 27\lambda_8^2\lambda_6^4 \\ &\quad + (108\lambda_{10}^2\lambda_4^5 - 72\lambda_{10}\lambda_8\lambda_6\lambda_4^4 + 16\lambda_{10}\lambda_6^3\lambda_4^3 + 16\lambda_8^3\lambda_4^4 - 4\lambda_8^2\lambda_6^2\lambda_4^3), \end{aligned}$$

and

$$\Gamma(\lambda) = \begin{pmatrix} 50\lambda_{10}\lambda_6 - 80\lambda_8^2 + 36\lambda_8\lambda_4^2 - 27\lambda_6^2\lambda_4 - 4\lambda_4^4 \\ 200\lambda_{10}\lambda_8 - 40\lambda_{10}\lambda_4^2 - 36\lambda_8\lambda_6\lambda_4 + 27\lambda_6^3 + 4\lambda_6\lambda_4^3 \\ 625\lambda_{10}^2 - 720\lambda_8^2\lambda_4 + 135\lambda_8\lambda_6^2 + 308\lambda_8\lambda_4^3 - 216\lambda_6^2\lambda_4^2 - 32\lambda_4^5 \\ 1600\lambda_8^3 - 1040\lambda_8^2\lambda_4^2 + 360\lambda_8\lambda_6^2\lambda_4 + 135\lambda_6^4 + 224\lambda_8\lambda_4^4 - 88\lambda_6^2\lambda_4^3 - 16\lambda_4^6 \end{pmatrix}.$$

Proof. Consider a curve (3) with at least one double point at $(x, y) = (a_2, 0)$. It has the form

$$(6) \quad -y^2 + (x - a_2)^2(x^3 + 2a_2x^2 + \mu_4x + \mu_6) = 0.$$

By subtracting (6) from (3) and collecting coefficients at the power of x we find the following polynomials in $(\lambda_{10}, \lambda_8, \lambda_6, \lambda_4; \mu_6, \mu_4, a_2)$:

$$(7) \quad \Upsilon(\lambda; \mu, a_2) = \begin{pmatrix} \lambda_4 - (\mu_4 - 3a_2^2) \\ \lambda_6 - (\mu_6 - 2a_2\mu_4 + 2a_2^3) \\ \lambda_8 - (-2a_2\mu_6 + a_2^2\mu_4) \\ \lambda_{10} - a_2^2\mu_6 \end{pmatrix}.$$

The polynomials vanish whenever a curve (3) has the form (6), that is the curve has genus not greater than 1, equivalently $\lambda \in \Lambda_1 \cup \Lambda_0$. The polynomials $\Upsilon(\lambda; \mu, a_2)$ generate an ideal $I_\Upsilon \subset \mathbb{C}[\lambda; \mu, a_2]$. Gröbner basis of $I_\Upsilon \cap \mathbb{C}[\lambda]$ is $\Delta(\lambda)$.

If $\delta(\mu, a_2) = 4(\mu_4 - \frac{4}{3}a_2^2)^3 + 27(\mu_6 - \frac{2}{3}a_2\mu_4 + \frac{16}{27}a_2^3)^2$ vanishes then the polynomial $x^3 + 2a_2x^2 + \mu_4x + \mu_6$, cf. (6), has a double root. This means the curve (6) has two double points and its genus is 0, equivalently $\lambda \in \Lambda_0$. The polynomials $\Upsilon(\lambda; \mu, a_2)$ and $\delta(\mu, a_2)$ generate an ideal $I_{(\Upsilon, \delta)} \subset \mathbb{C}[\lambda; \mu, a_2]$. Gröbner basis of $I_{(\Upsilon, \delta)} \cap \mathbb{C}[\lambda]$ is $\Gamma(\lambda)$.

To calculate Gröbner bases we use B. Buchberger's method with lexicographic monomial order. \square

Remark 1. The polynomial $\Delta(\lambda)$ is in fact the discriminant of $x^5 + \lambda_4x^3 + \lambda_6x^2 + \lambda_8x + \lambda_{10}$, cf. (3), while the polynomial $\delta(\mu, a_2)$ is the discriminant of $x^3 + 2a_2x^2 + \mu_4x + \mu_6$, cf. (6).

Introduce variables γ_4, γ_6 by the formulas $\gamma_4 = \mu_4 - \frac{4}{3}a_2^2$ and $\gamma_6 = \mu_6 - \frac{2}{3}a_2\mu_4 + \frac{16}{27}a_2^3$. Then the above polynomial $\delta(\mu, a_2)$ takes the form $\delta(\gamma) = 4\gamma_4^3 + 27\gamma_6^2$. In what follows we shall need the following expressions

$$(8) \quad \mu_4 = \gamma_4 + \frac{4}{3}a_2^2, \quad \mu_6 = \gamma_6 + \frac{2}{3}a_2\gamma_4 + \frac{8}{27}a_2^3.$$

Equations $\Upsilon(\lambda; \gamma, a_2) = 0$ with respect to (γ, a_2) , here μ in (7) are replaced by γ according to (8), have no solution when $\lambda \in \Lambda_2$, a unique solution for (γ, a_2) when $\lambda \in \Lambda_1$, and two solutions when $\lambda \in \Lambda_0$. Indeed, if $\Delta(\lambda) \neq 0$ the equations are incompatible. Let $\Delta(\lambda) = 0$, suppose there exist two distinct points (γ, a_2) and (β, b_2) corresponding to the same point λ . Subtracting $\Upsilon(\lambda; \beta, b_2) = 0$ from $\Upsilon(\lambda; \gamma, a_2) = 0$ then eliminating $\gamma_4 - \beta_4$ and $\gamma_6 - \beta_6$ we come to a pair of algebraic equations of order five and four with respect to $t = a_2 - b_2$. These equations have a single common root $t = 0$ iff $\delta(\gamma) \neq 0$, thus the points (γ, a_2) and (β, b_2) coincide. Now suppose both $\Delta(\lambda)$ and $\delta(\gamma)$ vanish, then $(\gamma_6, \gamma_4) = (2t^3, -3t^2)$ for some value of $t \in \mathbb{C}$. The system $\Upsilon(\lambda; (2t^3, -3t^2), a_2) - \Upsilon(\lambda; (2s^3, -3s^2), b_2) = 0$ is satisfied by two solutions: $(s, b_2) = (t, a_2)$ and $(s, b_2) = (\frac{2}{3}t + \frac{5}{9}a_2, t - \frac{2}{3}a_2)$.

3. FRAMES IN STRATA

To define a frame in the stratum Λ_2 we use a theorem due to V.M. Zakalyukin [30], see also [28], which puts into correspondence a vector field L tangent to hypersurface $\Delta(\lambda) = 0$ and a polynomial $p(x, y)$, namely,

$$Lf(x, y) = p(x, y)f(x, y) \mod (\partial_x f, \partial_y f),$$

where $f(x, y) = 0$ is a curve equation. In our case

$$\mathbb{C}[x, y]/(\partial_x f, \partial_y f) = \text{span}_{\mathbb{C}}(1, x, x^2, x^3),$$

and four vector fields $\{\ell_0, \ell_2, \ell_4, \ell_6\}$ correspondent to the polynomials

$$\begin{aligned} p_0(x, y) &= 10, & p_2(x, y) &= 10x, \\ p_4(x, y) &= 10x^2 + 6\lambda_4, & p_6(x, y) &= 10x^3 + 6\lambda_4x + 4\lambda_6 \end{aligned}$$

provide a basis in Λ_2 . Explicitly

$$(9) \quad \begin{pmatrix} \ell_0 \\ \ell_2 \\ \ell_4 \\ \ell_6 \end{pmatrix} = V(\lambda) \begin{pmatrix} \partial_{\lambda_4} \\ \partial_{\lambda_6} \\ \partial_{\lambda_8} \\ \partial_{\lambda_{10}} \end{pmatrix},$$

where

$$(10) \quad V(\lambda) = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_6\lambda_4 & -\frac{4}{5}\lambda_8\lambda_4 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_6\lambda_4 & 4\lambda_8\lambda_4 - \frac{12}{5}\lambda_6^2 & 6\lambda_{10}\lambda_4 - \frac{6}{5}\lambda_8\lambda_6 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_8\lambda_4 & 6\lambda_{10}\lambda_4 - \frac{6}{5}\lambda_8\lambda_6 & 4\lambda_{10}\lambda_6 - \frac{8}{5}\lambda_8^2 \end{pmatrix}.$$

Vector fields $\ell = (\ell_0, \ell_2, \ell_4, \ell_6)$ are tangent to discriminant variety $\{\lambda \mid \Delta(\lambda) = 0\} \cong \Lambda_1 \cup \Lambda_0$, in fact,

$$(11) \quad \ell_k \Delta(\lambda) = \phi_k \Delta(\lambda), \quad \phi_k \in \mathbb{C}[\lambda], \quad k = 0, 2, 4, 6;$$

$$\phi = (40, 0, 12\lambda_4, 4\lambda_6).$$

Vector fields ℓ are tangent to the variety $\{\lambda \mid \Gamma(\lambda) = 0\} \cong \Lambda_0$, namely

$$(12) \quad \ell_k \Gamma(\lambda) = \psi_k \Gamma(\lambda), \quad \psi_k \in \text{Mat}(4; \mathbb{C}[\lambda]), \quad k = 0, 2, 4, 6;$$

$$\psi_0 = \text{diag}(16, 18, 20, 24), \quad \psi_2 = \begin{pmatrix} 0 & -6 & 0 & 0 \\ -\frac{116}{5}\lambda_4 & 0 & \frac{16}{5} & 0 \\ 27\lambda_6 & -77\lambda_4 & 0 & 0 \\ 72\lambda_6\lambda_4 & 240\lambda_8 - 56\lambda_4^2 & 0 & 0 \end{pmatrix},$$

$$\psi_4 = \begin{pmatrix} -\frac{32}{5}\lambda_4 & 0 & \frac{4}{5} & 0 \\ \frac{33}{5}\lambda_6 & 5\lambda_4 & 0 & 0 \\ 24\lambda_8 - \frac{432}{5}\lambda_4^2 & 0 & 12\lambda_4 & -\frac{12}{5} \\ 144\lambda_8\lambda_4 + 108\lambda_6^2 - \frac{176}{5}\lambda_4^3 & 0 & 0 & \frac{44}{5}\lambda_4 \end{pmatrix},$$

$$\psi_6 = \begin{pmatrix} -\frac{7}{5}\lambda_6 & -\frac{7}{5}\lambda_4 & 0 & 0 \\ 4\lambda_8 - \frac{128}{25}\lambda_4^2 & 0 & \frac{16}{25}\lambda_4 & 0 \\ 100\lambda_{10} - \frac{81}{5}\lambda_6\lambda_4 & -6\lambda_8 - \frac{81}{5}\lambda_4^2 & 0 & 0 \\ 72\lambda_8\lambda_6 - \frac{48}{5}\lambda_6\lambda_4^2 & 40\lambda_8\lambda_4 - \frac{48}{5}\lambda_4^3 & 0 & 0 \end{pmatrix}.$$

It follows from $\det V(\lambda) = \frac{16}{5}\Delta(\lambda)$ that ℓ defines a frame in the stratum Λ_2 ; next (11) and (12) imply that restrictions of ℓ to the strata Λ_1 and Λ_0 provide frames on the both strata. To analyze the restrictions in more detail we need parameterization of Λ_1 and Λ_0 . By combining (6) with (8) and comparing with (3) we observe that the subset of curves (3) with one double point is parameterized as follows

$$(13) \quad \begin{aligned} \lambda_4 &= \gamma_4 - \frac{5}{3}a_2^2, \\ \lambda_6 &= \gamma_6 - \frac{4}{3}a_2\gamma_4 - \frac{10}{27}a_2^3, \\ \lambda_8 &= -2a_2\gamma_6 - \frac{1}{3}a_2^2\gamma_4 + \frac{20}{27}a_2^4, \\ \lambda_{10} &= a_2^2\gamma_6 + \frac{2}{3}a_2^3\gamma_4 + \frac{8}{27}a_2^5, \\ 4\gamma_4^3 + 27\gamma_6^2 &\neq 0. \end{aligned}$$

Lemma 1. *The restricted vector fields $(\tilde{\ell}_0, \tilde{\ell}_2, \tilde{\ell}_4) = (\ell_0, \ell_2, \ell_4)|_{\Lambda_1}$ form a frame on the stratum Λ_1 . In terms of parameterization (13) they are expressed as follows*

$$\begin{aligned} \tilde{\ell}_0 &= 2a_2\partial_{a_2} + 4\gamma_4\partial_{\gamma_4} + 6\gamma_6\partial_{\gamma_6}, \\ \tilde{\ell}_2 &= \frac{2}{15}(6\gamma_4 + 5a_2^2)\partial_{a_2} + \frac{2}{3}(9\gamma_6 - 8a_2\gamma_4)\partial_{\gamma_4} - \frac{4}{3}(\gamma_4^2 + 6a_2\gamma_6)\partial_{\gamma_6}, \\ \tilde{\ell}_4 &= \frac{2}{45}(27\gamma_6 + 9a_2\gamma_4 - 40a_2^3)\partial_{a_2} - \frac{4}{3}a_2(9\gamma_6 + a_2\gamma_4)\partial_{\gamma_4} - \frac{2}{3}a_2(3a_2\gamma_6 - 4\gamma_4^2)\partial_{\gamma_6}. \end{aligned}$$

On Λ_1 the vector field ℓ_6 is decomposed into

$$(14) \quad \ell_6|_{\Lambda_1} = -a_2^3 \tilde{\ell}_0 - a_2^2 \tilde{\ell}_2 - a_2 \tilde{\ell}_4.$$

Proof. The proof is straightforward. \square

Remark 2. The vector fields $(\tilde{\ell}_0, \tilde{\ell}_2, \tilde{\ell}_4)$ on a curve (3) with a double point at $(a_2, 0)$ can be expressed in terms of the three vector fields: ∂_{a_2} , $L_0 = 4\gamma_4\partial_{\gamma_4} + 6\gamma_6\partial_{\gamma_6}$, and $L_2 = 6\gamma_6\partial_{\gamma_4} - \frac{4}{3}\gamma_4^2\partial_{\gamma_6}$ as follows

$$(15) \quad \begin{aligned} \tilde{\ell}_0 &= 2a_2\partial_{a_2} + L_0, \\ \tilde{\ell}_2 &= \frac{2}{15}(6\gamma_4 + 5a_2^2)\partial_{a_2} - \frac{4}{3}a_2L_0 + L_2, \\ \tilde{\ell}_4 &= \frac{2}{45}(27\gamma_6 + 9a_2\gamma_4 - 40a_2^3)\partial_{a_2} - \frac{1}{3}a_2^2L_0 - 2a_2L_2. \end{aligned}$$

The fields L_0, L_2 are tangent to the variety $\{\gamma \mid \delta(\gamma) = 0\}$.

In a similar way, from the generic form of a curve (3) with two double points at $(a_2, 0)$ and $(b_2, 0)$

$$(16) \quad -y^2 + (x - a_2)^2(x - b_2)^2(x + 2a_2 + 2b_2) = 0$$

we obtain a parameterization of Λ_0

$$(17) \quad \begin{aligned} \lambda_4 &= -3a_2^2 - 4a_2b_2 - 3b_2^2, \\ \lambda_6 &= 2(a_2 + b_2)(a_2^2 + 3a_2b_2 + b_2^2), \\ \lambda_8 &= -a_2b_2(4a_2^2 + 7b_2a_2 + 4b_2^2), \\ \lambda_{10} &= 2a_2^2b_2^2(a_2 + b_2). \end{aligned}$$

Lemma 2. The restricted vector fields $(\tilde{\ell}_0, \tilde{\ell}_2) = (\ell_0, \ell_2)|_{\Lambda_0}$ form a frame on the stratum Λ_0 . In terms of parameterization (17) they are expressed as follows

$$\begin{aligned} \tilde{\ell}_0 &= 2a_2\partial_{a_2} + 2b_2\partial_{b_2}, \\ \tilde{\ell}_2 &= -\frac{2}{5}(a_2^2 + 8a_2b_2 + 6b_2^2)\partial_{a_2} - \frac{2}{5}(6a_2^2 + 8a_2b_2 + b_2^2)\partial_{b_2}. \end{aligned}$$

On Λ_0 the vector fields ℓ_4 and ℓ_6 are decomposed into

$$(18a) \quad \ell_4|_{\Lambda_0} = -(a_2^2 + a_2b_2 + b_2^2)\tilde{\ell}_0 - (a_2 + b_2)\tilde{\ell}_2$$

$$(18b) \quad \ell_6|_{\Lambda_0} = a_2b_2(a_2 + b_2)\tilde{\ell}_0 + a_2b_2\tilde{\ell}_2.$$

Proof. The proof is straightforward. \square

4. ANNIHILATORS OF SIGMA-FUNCTION

Following [11], we write down the operators producing heat equations in a non-holonomic frame in the case of genus 2 curve (3):

$$\begin{aligned} q_0 &= -u_1\partial_{u_1} - 3u_3\partial_{u_3} + 3 + \ell_0, \\ q_2 &= -\frac{1}{2}\partial_{u_1u_1} + \frac{4}{5}\lambda_4u_3\partial_{u_1} - u_1\partial_{u_3} + \frac{3}{10}\lambda_4u_1^2 - \frac{1}{10}(15\lambda_8 - 4\lambda_4^2)u_3^2 + \ell_2, \\ q_4 &= -\partial_{u_1u_3} + \frac{6}{5}\lambda_6u_3\partial_{u_1} - \lambda_4u_3\partial_{u_3} + \frac{1}{5}\lambda_6u_1^2 - \lambda_8u_1u_3 \\ &\quad - \frac{1}{10}(30\lambda_{10} - 6\lambda_6\lambda_4)u_3^2 + \lambda_4 + \ell_4, \\ q_6 &= -\frac{1}{2}\partial_{u_3u_3} + \frac{3}{5}\lambda_8u_3\partial_{u_1} + \frac{1}{10}\lambda_8u_1^2 - 2\lambda_{10}u_1u_3 + \frac{3}{10}\lambda_8\lambda_4u_3^2 \\ &\quad + \frac{1}{2}\lambda_6 + \ell_6. \end{aligned}$$

We define sigma-function $\sigma(u_3, u_1; \lambda)$ on genus 2 curve (3) as a solution of the equations

$$q_k\sigma(u_3, u_1; \lambda) = 0, \quad k = 0, 2, 4, 6$$

with the initial condition $\sigma(u_3, 0; 0) = u_3$. Since the solution is unique [11], this completely defines the sigma-function.

According to relation (14) from Lemma 1 the operator $Q_6 = -2(q_6 + a_2 q_4 + a_2^2 q_2 + a_2^3 q_0)|_{\Lambda_1}$ does not include derivatives over γ and a_2 , namely:

$$(20) \quad Q_6 = \left(\partial_{u_3} + a_2 \partial_{u_1} + a_2^2 u_1 + \left(\gamma_4 + \frac{4}{3} a_2^2 \right) a_2 u_3 \right)^2 - \left(\gamma_6 + \frac{5}{3} a_2 \gamma_4 + \frac{125}{27} a_2^3 \right).$$

Introduce a new variable U_1 by the formula $u_1 = U_1 + a_2 u_3$, then (20) becomes an ordinary differential operator

$$(21) \quad Q_6 = D^2 - d(a_2, \gamma)^2$$

with the operator D and the function $d(a_2, \gamma)$ given by

$$D = \partial_{u_3} + a_2^2 U_1 + \left(\gamma_4 + \frac{7}{3} a_2^2 \right) a_2 u_3 \quad d(a_2, \gamma)^2 = \gamma_6 + \frac{5}{3} a_2 \gamma_4 + \left(\frac{5}{3} a_2 \right)^3.$$

The operator $Q_4 = -(q_4 + 2a_2 q_2 + 3a_2^2 q_0)|_{\Lambda_1}$ has the form

$$(22) \quad Q_4 = \left(\partial_{U_1} + 2a_2 U_1 + \left(\gamma_4 + \frac{28}{3} a_2^2 \right) u_3 \right) D - \frac{6}{5} d(a_2, \gamma) \partial_{a_2} (d(a_2, \gamma) \cdot) \\ - \frac{1}{5} (U_1^2 + 12a_2 U_1 u_3 + 3(\gamma_4 + 7a_2^2) u_3^2) d(a_2, \gamma)^2.$$

Then $Q_0 = q_0|_{\Lambda_1}$ and $Q_2 = q_2 + \frac{4}{3} a_2 q_0|_{\Lambda_1}$ take the form

$$(23a) \quad Q_0 = -U_1 \partial_{U_1} - 3u_3 \partial_{u_3} + 2a_2 \partial_{a_2} + L_0 + 3,$$

$$(23b) \quad Q_2 = -\frac{1}{2} \partial_{U_1 U_1} - \frac{1}{3} a_2 (U_1 + 3a_2 u_3) \partial_{U_1} - (U_1 + 5a_2 u_3) \partial_{u_3} \\ + \frac{2}{15} (6\gamma_4 + 25a_2^2) \partial_{a_2} + L_2 + \frac{1}{10} (3\gamma_4 - 5a_2^2) (U_1 + 2a_2 u_3) U_1 \\ + \frac{1}{30} (90a_2 \gamma_6 + 12\gamma_4^2 - 16a_2^2 \gamma_4 - 15a_2^4) u_3^2 + 4a_2.$$

A solution $\mathcal{Z}(u_3, U_1, a_2, \gamma)$ of the system

$$Q_k \mathcal{Z} = 0, \quad k = 0, 2, 4, 6, \quad \mathcal{Z}(u_3, 0, 0, 0) = u_3$$

at $U_1 = u_1 - a_2 u_3$ is a degenerate sigma-function and coincides with $\sigma(u_3, u_1; \lambda)$ restricted to Λ_1 . We construct this solution explicitly in the next section.

5. DEGENERATE SIGMA-FUNCTION

Theorem 1. Suppose $\lambda \in \Lambda_1$. Sigma-function associated with a curve (3) has the form

$$(24) \quad \sigma(u_3, u_1, \lambda)|_{\Lambda_1} = \frac{e^{-\frac{3}{5} \wp(\alpha) \left(\left(\frac{1}{2} \gamma_4 + \frac{3}{25} \wp(\alpha)^2 \right) u_3^2 + \frac{2}{5} \wp(\alpha) u_1 u_3 + \frac{1}{6} u_1^2 \right)}}{\wp'(\alpha) \sigma(\alpha)} \times \\ \times \left(\sigma\left(\alpha + u_1 - \frac{3}{5} \wp(\alpha) u_3\right) e^{\frac{1}{2} \wp'(\alpha) u_3 - \zeta(\alpha) \left(u_1 - \frac{3}{5} \wp(\alpha) u_3 \right)} \right. \\ \left. - \sigma\left(\alpha - u_1 + \frac{3}{5} \wp(\alpha) u_3\right) e^{-\frac{1}{2} \wp'(\alpha) u_3 + \zeta(\alpha) \left(u_1 - \frac{3}{5} \wp(\alpha) u_3 \right)} \right),$$

where σ , ζ , \wp are Weierstrass functions associated with the curve (4), and α is defined by $\wp(\alpha) = \frac{5}{3} a_2$.

Proof. First, we consider the equation

$$Q_6 \mathcal{Z}(u_3, U_1, a_2, \gamma) = 0$$

where Q_6 is defined by (21). The gauge transformation

$$(25) \quad \mathcal{Z}(u_3, U_1, a_2, \gamma) = \exp \left\{ -\frac{1}{2} a_2 \left(\gamma_4 + \frac{7}{3} a_2^2 \right) u_3^2 - a_2^2 U_1 u_3 \right\} \rho(u_3, U_1, a_2, \gamma)$$

leads to a simpler equation

$$\partial_{u_3 u_3} \rho(u_3, U_1, a_2, \gamma) - d(a_2, \gamma)^2 \rho(u_3, U_1, a_2, \gamma) = 0.$$

As fundamental solutions of the equation we choose $c_\epsilon(U_1, a_2, \gamma) \exp(\epsilon d(a_2, \gamma) u_3)$, where ϵ is unary operator: $\epsilon = \pm$. Then

$$(26) \quad \rho(u_3, U_1, a_2, \gamma) = c_+(U_1, a_2, \gamma) e^{u_3 d(a_2, \gamma)} + c_-(U_1, a_2, \gamma) e^{-u_3 d(a_2, \gamma)}.$$

Next, consider the equation

$$Q_4 \mathcal{Z}(u_3, U_1, a_2, \gamma) = 0.$$

Taking into account (25) and (26) we obtain the following equations for c_ϵ

$$\epsilon \partial_{U_1} c_\epsilon - \frac{6}{5} \partial_{a_2} (d(a_2, \gamma) c_\epsilon) = \left(-2\epsilon a_2 U_1 + \frac{1}{5} d(a_2, \gamma) U_1^2 \right) c_\epsilon.$$

The substitution

$$c_\epsilon(U_1, a_2, \gamma) = \exp \{ \varphi_\epsilon(U_1, a_2, \gamma) \} / d(a_2, \gamma)$$

leads to a linear non-homogeneous PDE

$$(27) \quad (\epsilon \partial_{U_1} - \frac{6}{5} d(a_2, \gamma) \partial_{a_2}) \varphi_\epsilon = -2\epsilon a_2 U_1 + \frac{1}{5} d(a_2, \gamma) U_1^2.$$

We solve the associated homogeneous equation by the method of characteristics:

$$-\epsilon dU_1 = \frac{5}{6} \frac{da_2}{d(a_2, \gamma)} = \frac{d(\frac{5}{3} a_2)}{-2\sqrt{(\frac{5}{3} a_2)^3 + \frac{5}{3} a_2 \gamma_4 + \gamma_6}}.$$

The characteristics is defined by the equation

$$\alpha(a_2, \gamma) + \epsilon U_1 = \text{const},$$

where

$$\alpha(a_2, \gamma) = \int_{\infty}^{\frac{5}{3} a_2} \frac{dX}{-2\sqrt{X^3 + \gamma_4 X + \gamma_6}}, \quad \deg \alpha = 1.$$

We write down a general solution of the homogeneous equation as

$$\varphi_\epsilon^{(h)} = \log s_\epsilon(\alpha(a_2, \gamma) + \epsilon U_1, \gamma).$$

In what follows we need elliptic functions σ , ζ , \wp , \wp' associated with the curve (4). Here

$$\wp(\alpha) = \frac{5}{3} a_2 \quad \wp'(\alpha) = 2d(a_2, \gamma) = -2\sqrt{\left(\frac{5}{3} a_2\right)^3 + \frac{5}{3} a_2 \gamma_4 + \gamma_6}.$$

The functions \wp , \wp' satisfy the equation $(\wp')^2 = 4\wp^3 + 4\lambda_4 \wp + 4\lambda_6$, thus, they are standard Weierstrass functions with the invariants $(g_2, g_3) = (-4\lambda_4, -4\lambda_6)$, see [23].

Next, we construct a particular solution of non-homogeneous equation (27) in the form

$$\varphi_\epsilon^{(nh)} = C_2(a_2, \gamma) U_1^2 + C_1(a_2, \gamma) U_1 + C_0(a_2, \gamma).$$

By substituting the ansatz and collecting coefficients at the powers of U_1 we obtain a system of equations for C_2 , C_1 and C_0 :

$$\partial_{a_2} C_2 = -\frac{1}{6}, \quad \frac{6}{5} d(a_2, \gamma) \partial_{a_2} C_1 = 2\epsilon(C_2 + a_2), \quad \frac{6}{5} d(a_2, \gamma) \partial_{a_2} C_0 = \epsilon C_1.$$

Observe that $\partial_\alpha = \frac{6}{5} d(a_2, \gamma) \partial_{a_2}$. Whence

$$\begin{aligned} C_2(a_2, \gamma) &= -\frac{1}{6} a_2, \\ \partial_\alpha C_1 &= \epsilon \frac{5}{3} a_2 = \epsilon \wp(\alpha) &\Rightarrow C_1(a_2, \gamma) &= -\epsilon \zeta(\alpha), \\ \partial_\alpha C_0 &= -\zeta(\alpha) &\Rightarrow C_0(a_2, \gamma) &= -\log \sigma(\alpha), \end{aligned}$$

above we have used the standard relations:

$$\zeta(u) = -\int_{\infty}^u \wp(v) dv, \quad \log \sigma(u) = \int_{\infty}^u \zeta(v) dv.$$

Summing up, the general solution $\varphi_\epsilon^{(h)} + \varphi_\epsilon^{(nh)}$ of (27) has the form

$$\varphi_\epsilon(U_1, a_2, \gamma) = \log s_\epsilon(\alpha(a_2, \gamma) + \epsilon U_1, \gamma) - \frac{1}{6}a_2 U_1^2 - \epsilon \zeta(\alpha) U_1 - \log \sigma(\alpha).$$

Therefore, we come to the following expression for c_ϵ :

$$(28) \quad c_\epsilon(U_1, a_2, \gamma) = \frac{s_\epsilon(\alpha(a_2, \gamma) + \epsilon U_1, \gamma)}{\wp'(\alpha(a_2, \gamma))\sigma(\alpha(a_2, \gamma))} e^{-\frac{1}{6}a_2 U_1^2 - \epsilon \zeta(\alpha(a_2, \gamma)) U_1}.$$

Taking into account the form (28) of dependence of c_ϵ on a_2 , for the next step we change the variables on Λ_1 from $(a_2, \gamma_4, \gamma_6)$ to $(\alpha, \gamma_4, \gamma_6)$:

$$c_\epsilon(U_1, \alpha, \gamma) = \frac{s_\epsilon(\alpha + \epsilon U_1, \gamma)}{\wp'(\alpha)\sigma(\alpha)} e^{-\frac{1}{10}\wp(\alpha)U_1^2 - \epsilon \zeta(\alpha)U_1}.$$

Under the change of variables the operators Q_2, Q_0 map to new operators \tilde{Q}_2, \tilde{Q}_0 , where the map is defined by the following formula (cf. Remark 2)

$$(\partial_{a_2}, L_2, L_0) \mapsto \left(\frac{5}{3\wp'(\alpha)} \partial_\alpha, L_2 - \frac{L_2(\wp(\alpha))}{\wp'(\alpha)} \partial_\alpha, L_0 - \frac{L_0(\wp(\alpha))}{\wp'(\alpha)} \partial_\alpha \right).$$

Applying the operator \tilde{Q}_2 to $\mathcal{Z}(u_3, U_1, \frac{3}{5}\wp(\alpha), \gamma)$ with ansatz (28) and using the relations

$$\begin{aligned} L_2 \sigma(\alpha) &= \sigma(\alpha) \left(-\frac{1}{6} \gamma_4 \alpha^2 + \frac{1}{2} \zeta(\alpha)^2 - \frac{1}{2} \wp(\alpha) \right), \\ L_2 \zeta(\alpha) &= -\frac{1}{3} \gamma_4 \alpha - \zeta(\alpha) \wp(\alpha) - \frac{1}{2} \wp'(\alpha), \\ L_2 \wp(\alpha) &= \frac{4}{3} \gamma_4 + 2 \wp(\alpha)^2 + \zeta(\alpha) \wp'(\alpha), \\ L_2 \wp'(\alpha) &= \zeta(\alpha) (6 \wp(\alpha)^2 + 2 \gamma_4) + 3 \wp(\alpha) \wp'(\alpha). \end{aligned}$$

we come to the equation

$$(29) \quad \left(-\frac{1}{2} \partial_{U_1} U_1 + \frac{1}{6} \gamma_4 (\alpha + \epsilon U_1)^2 + L_2 \right) s_\epsilon(\alpha + \epsilon U_1, \gamma) = 0.$$

Similarly, the operator \tilde{Q}_0 leads to the equation

$$(30) \quad \left(-(\alpha + \epsilon U_1) \partial_{U_1} + L_0 + 1 \right) s_\epsilon(\alpha + \epsilon U_1, \gamma) = 0.$$

Further, consider the power series expansion for $\mathcal{Z}(u_3, 0, \frac{3}{5}\wp(\alpha), \gamma)$ in u_3 near zero. We obtain

$$\mathcal{Z}(u_3, 0, \frac{3}{5}\wp(\alpha), \gamma) = \frac{s_+(\alpha, \gamma) + s_-(\alpha, \gamma)}{\sigma(\alpha)\wp'(\alpha)} + \frac{s_+(\alpha, \gamma) - s_-(\alpha, \gamma)}{2\sigma(\alpha)} u_3 + O(u_3^2).$$

Comparing the expansion with the initial condition $\mathcal{Z}(u_3, 0, 0, 0) = u_3$ for entire function \mathcal{Z} and taking into account that at $\gamma = (0, 0)$ the value of $\alpha(a_2, \gamma)$ tends to infinity as $a_2 \rightarrow 0$ we find

$$\begin{aligned} s_+(\alpha, 0) &= -s_-(\alpha, 0), \\ s_+(\alpha, 0) &= \sigma(\alpha)|_{\gamma=0} = \alpha. \end{aligned}$$

Therefore, $s_\epsilon(\alpha, 0) = \epsilon \alpha$. Thus, the initial condition singles out a unique solution of equations (29) and (30) that is

$$s_\epsilon(\alpha + \epsilon U_1, \gamma) = \epsilon \sigma(\alpha + \epsilon U_1).$$

Combining all of the above results we write down the final expression for \mathcal{Z} . \square

Remark 3. Note that the genus 2 degenerate sigma-function (24) can be represented with the help of elliptic Baker function Φ

$$(31) \quad \Phi(u, \alpha) = \frac{\sigma(\alpha - u)}{\sigma(\alpha)\sigma(u)} e^{\zeta(\alpha)u}.$$

Indeed, we have

$$\begin{aligned} \sigma(u_3, u_1, \lambda)|_{\Lambda_1} = & -e^{-\frac{3}{5}\wp(\alpha)} \left(\left(\frac{1}{2}\gamma_4 + \frac{3}{25}\wp(\alpha)^2 \right) u_3^2 + \frac{2}{5}\wp(\alpha)u_1u_3 + \frac{1}{6}u_1^2 \right) \frac{\sigma(u_1 - \frac{3}{5}\wp(\alpha)u_3)}{\wp'(\alpha)} \times \\ & \times \left(\Phi(-u_1 + \frac{3}{5}\wp(\alpha)u_3, \alpha) e^{\frac{1}{2}\wp'(\alpha)u_3} + \Phi(u_1 - \frac{3}{5}\wp(\alpha)u_3, \alpha) e^{-\frac{1}{2}\wp'(\alpha)u_3} \right). \end{aligned}$$

Remark 4. Visibly right hand side of (24) is singular when $\sigma(\alpha) = 0$ or $\wp'(\alpha) = 0$. The first case corresponds to $a_2 = \infty$ which does not belong to Λ_1 , otherwise the equation (3) would not include the term x^5 . In the second case 2α is a period, say ω_1 , of Weierstrass functions, that is $\frac{5}{3}a_2$ becomes a branch point e_i of (4). Then

$$\begin{aligned} \sigma(u_3, u_1, \lambda)|_{\Lambda_1} = & u_3 e^{-\frac{3}{5}\wp(\alpha)} \left(\left(\frac{1}{2}\gamma_4 + \frac{3}{25}\wp(\alpha)^2 \right) u_3^2 + \frac{2}{5}\wp(\alpha)u_1u_3 + \frac{1}{6}u_1^2 \right) \times \\ (32) \quad & \times \frac{\sigma(\alpha + u_1 - \frac{3}{5}\wp(\alpha)u_3)}{\sigma(\alpha)} e^{-\zeta(\alpha)(u_1 - \frac{3}{5}\wp(\alpha)u_3)} \\ & = u_3 e^{-\frac{3}{5}e_i} \left(\left(\frac{1}{2}\gamma_4 + \frac{3}{25}e_i^2 \right) u_3^2 + \frac{2}{5}e_i u_1u_3 + \frac{1}{6}u_1^2 \right) \sigma_i(u_1 - \frac{3}{5}e_1u_3), \end{aligned}$$

where $\sigma_i(u) = \exp(-u\eta_i)\sigma(u + \omega_i)/\sigma(\omega_i)$ with $e_i = \wp(\omega_i)$, $\eta_i = \zeta(\omega_i)$ denotes a sigma-function with characteristic [23], p.348 eq.(22).

Theorem 2. Suppose $\lambda \in \Lambda_0$. Sigma-function associated with a curve (3) has the form

$$\begin{aligned} (33) \quad \sigma(u_3, u_1, \lambda)|_{\Lambda_0} = & \frac{e^{\frac{1}{2}(3a_2b_2(a_2+b_2)u_3^2 + 2a_2b_2u_1u_3 - (a_2+b_2)u_1^2)}}{4(a_2 - b_2)} \times \\ & \times \left(\cosh(\sqrt{2a_2 + 3b_2}(u_1 - a_2u_3)) \frac{\sinh(\sqrt{3a_2 + 2b_2}(u_1 - b_2u_3))}{\sqrt{3a_2 + 2b_2}} \right. \\ & \left. - \cosh(\sqrt{3a_2 + 2b_2}(u_1 - b_2u_3)) \frac{\sinh(\sqrt{2a_2 + 3b_2}(u_1 - a_2u_3))}{\sqrt{2a_2 + 3b_2}} \right). \end{aligned}$$

The theorem is proven by an argument similar to the proof of Theorem 1.

6. APPLICATIONS

6.1. A generalized Jacobi inversion problem. Let (X_1, Y_1) and (X_2, Y_2) be a pair of points on the elliptic curve (4). Consider an inversion problem for integrals

$$\begin{aligned} (34) \quad & \int_{\infty}^{(X_1, Y_1)} \frac{dX}{-2Y} + \int_{\infty}^{(X_2, Y_2)} \frac{dX}{-2Y} = U_1, \\ & \int_{\infty}^{(X_1, Y_1)} \frac{dX}{-2Y(X-A)} + \int_{\infty}^{(X_2, Y_2)} \frac{dX}{-2Y(X-A)} = U_3. \end{aligned}$$

Denote

$$\mathcal{Z} = \sigma(U_3, U_1 + \frac{3}{5}AU_3, \lambda(A, \gamma)),$$

and $\lambda(A, \gamma)$ is defined by

$$\begin{aligned} (35) \quad & \lambda_4 = \gamma_4 - \frac{3}{5}A^2, \\ & \lambda_6 = \gamma_6 - \frac{4}{5}A\gamma_4 - \frac{2}{25}A^3, \\ & \lambda_8 = -\frac{6}{5}A\gamma_6 - \frac{3}{25}A^2\gamma_4 + \frac{12}{125}A^4, \\ & \lambda_{10} = \frac{9}{25}A^2\gamma_6 + \frac{18}{125}A^3\gamma_4 + \frac{72}{3125}A^5. \end{aligned}$$

Further, let

$$\mathcal{P}_{ij} = -\partial_{U_i U_j} \log \mathcal{Z} \quad \text{and} \quad \mathcal{P}_{ijk} = -\partial_{U_i U_j U_k} \log \mathcal{Z}.$$

Corollary 1. *The solution of a generalized Jacobi inversion problem (34) is given by the formulas*

$$\begin{aligned} X_1 + X_2 &= \mathcal{P}_{11} + \frac{4}{5}A, \\ X_1 X_2 &= -\mathcal{P}_{13} + A\mathcal{P}_{11} + \frac{4}{25}A^2, \\ Y_k &= -\frac{1}{2}\mathcal{P}_{111} - \frac{\mathcal{P}_{113}}{2(X_k - A)}, \quad k = 1, 2. \end{aligned} \quad (36)$$

Proof. Consider the Jacobi inversion problem on a genus 2 curve of the form (3)

$$\begin{aligned} \int_{P_0}^{(x_1, y_1)} \frac{dx}{-2y} + \int_{P_0}^{(x_2, y_2)} \frac{dx}{-2y} &= u_3, \\ \int_{P_0}^{(x_1, y_1)} \frac{x dx}{-2y} + \int_{P_0}^{(x_2, y_2)} \frac{x dx}{-2y} &= u_1. \end{aligned} \quad (37)$$

The pair of points (x_1, y_1) and (x_2, y_2) on the curve is defined by formulas

$$\begin{aligned} x_1 + x_2 &= \wp_{11}, \quad x_1 x_2 = -\wp_{13}, \\ y_k &= -\frac{1}{2}(x_k \wp_{111} + \wp_{113}), \quad k = 1, 2, \end{aligned} \quad (38)$$

where $\wp_{ij} = -\partial_{u_i u_j} \log \sigma(u_3, u_1, \lambda)$ and $\wp_{ijk} = -\partial_{u_i u_j u_k} \log \sigma(u_3, u_1, \lambda)$. For more details see [3].

Indeed, relations (38) hold for all values of u and λ where sigma-function does not vanish. Consider (37) with parameters λ as in (35). The substitution

$$\begin{aligned} x &= X - \frac{2}{5}A, \quad y = Y(X - A), \\ u_3 &= U_3, \quad u_1 = U_1 + \frac{3}{5}AU_3 \end{aligned} \quad (39)$$

transforms the problem (37) to the problem (34). Consequently, (38) transforms to (36). \square

Introducing the following notation

$$\mathcal{P} = \frac{\sigma(\alpha + U_1)}{\sigma(\alpha - U_1)} e^{\wp'(\alpha)U_3 - 2\zeta(\alpha)U_1}, \quad (40a)$$

$$\mathcal{S} = \frac{1}{2(\wp(U_1) - \wp(\alpha))} \left(\wp'(U_1) - \wp'(\alpha) \frac{\mathcal{P} + 1}{\mathcal{P} - 1} \right), \quad (40b)$$

where $\wp(\alpha) = A$, we present explicit expressions for (36):

$$X_1 + X_2 = \mathcal{S}^2 - \wp(U_1), \quad (41a)$$

$$X_1 X_2 = \wp(U_1) \mathcal{S}^2 - \wp'(U_1) \mathcal{S} - \wp(\alpha) (\wp(U_1) + \wp(\alpha)) + \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{4(\wp(U_1) - \wp(\alpha))}, \quad (41b)$$

and from $Y_k = -\frac{1}{2} (X_k \partial_{U_1} (X_1 + X_2) - \partial_{U_1} (X_1 X_2)) / (X_k - \wp(\alpha))$

$$\begin{aligned} Y_k &= -\frac{X_k - \wp(U_1)}{X_k - \wp(\alpha)} \mathcal{S}^3 + \frac{\wp'(U_1)}{X_k - \wp(\alpha)} \mathcal{S}^2 \\ &\quad + \left(2\wp(U_1) + \wp(\alpha) + \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{4(X_k - \wp(\alpha))(\wp(U_1) - \wp(\alpha))} \right) \mathcal{S} + \frac{1}{2} \wp'(U_1). \end{aligned} \quad (41c)$$

Example 1. In the case when A is a branch point, say $e_1 = \wp(\omega/2)$, of the curve (4) the function σ is simplified dramatically, cf. (32). However formula (41c) fails for one of the roots. The explicit solution has the form

$$(42) \quad (X_1, Y_1) = (e_1, 0), \quad (X_2, Y_2) = (\wp(U_1 + \omega/2), -\frac{1}{2}\wp'(U_1 + \omega/2)).$$

Introducing variables ξ_k by the equalities $\wp(\xi_k) = X_k$, $k = 1, 2$ we rewrite the problem (34) in the form

$$(43) \quad \begin{aligned} \xi_1 + \xi_2 &= U_1, \\ \int_0^{\xi_1} \frac{d\xi}{\wp(\xi) - \wp(\alpha)} + \int_0^{\xi_2} \frac{d\xi}{\wp(\xi) - \wp(\alpha)} &= U_3. \end{aligned}$$

With the help of

$$\frac{\wp'(\alpha)}{\wp(\alpha) - \wp(\xi)} = 2\zeta(\alpha) - \zeta(\alpha - \xi) - \zeta(\alpha + \xi)$$

we explicitly integrate and reduce (43) to the following system

$$(44) \quad \xi_1 + \xi_2 = U_1, \quad \frac{\sigma(\alpha - \xi_1)\sigma(\alpha - \xi_2)}{\sigma(\alpha + \xi_1)\sigma(\alpha + \xi_2)} = e^{-2\zeta(\alpha)U_1 + \wp'(\alpha)U_3}.$$

Example 2. For the rational limit $(\gamma_4, \gamma_6) = 0$

$$\sigma(\xi) = \xi, \quad \zeta(\xi) = \xi^{-1}, \quad \wp(\xi) = \xi^{-2}, \quad \wp'(\xi) = -2\xi^{-3}$$

the problem (34) with usage of (44) is solved explicitly:

$$\xi_1 + \xi_2 = U_1, \quad \xi_1\xi_2 = -\alpha^2 + \alpha U_1 \left[\tanh\left(\frac{U_3}{\alpha^3} + \frac{U_1}{\alpha}\right) \right]^{-1}.$$

In the rational limit the same relations are obtained from (41) with $X_k = \xi_k^{-2}$.

Consider the equation with respect to ξ

$$(45) \quad e^{\varkappa} = \frac{\sigma(\xi - \alpha)\sigma(\xi + \beta)}{\sigma(\xi + \alpha)\sigma(\xi - 2\alpha + \beta)}.$$

Similar equations appear in the theory of Bethe ansatz, see [27] and many other publications. By combining the substitutions

$$\beta = \alpha - U_1, \quad \varkappa = -2\zeta(\alpha)U_1 + \wp'(\alpha)U_3.$$

the equation (45) is reduced to (44) and has two solutions ξ_1, ξ_2 defined by (41).

Remark 5. The ratio $\frac{\sigma(\xi - \alpha)\sigma(\xi + \beta)}{\sigma(\xi + \alpha)\sigma(\xi - 2\alpha + \beta)}$ can be represented as a rational function in $\wp(\xi)$ and $\wp'(\xi)$. Then the equation (45) is transformed to

$$e^{\varkappa} = \frac{\sigma(\beta)}{\sigma(2\alpha - \beta)} \cdot \frac{\wp(\alpha) - \wp(2\alpha - \beta)}{\wp(\alpha) - \wp(\beta)} \cdot \frac{\begin{vmatrix} \wp'(\xi) & \wp(\xi) & 1 \\ \wp'(\alpha) & \wp(\alpha) & 1 \\ -\wp'(\beta) & \wp(\beta) & 1 \end{vmatrix}}{\begin{vmatrix} \wp'(\xi) & \wp(\xi) & 1 \\ -\wp'(\alpha) & \wp(\alpha) & 1 \\ \wp'(2\alpha - \beta) & \wp(2\alpha - \beta) & 1 \end{vmatrix}}.$$

This is equivalent to an equation of the form $A\wp'(\xi) + B\wp(\xi) + C = 0$, which apparently has three roots. Two of the roots are functions in \varkappa and provide a solution of (45) and in fact are the same as defined by (41). The extra root $(\wp(\xi), \wp'(\xi)) = (\wp(\alpha - \beta), -\wp'(\alpha - \beta))$ is independent of \varkappa .

6.2. Schrödinger equation with periodic potential. Introduce the function

$$(46) \quad \Phi((u_3, u_1), (\beta_3, \beta_1)) = \frac{\sigma(\beta_3 - u_3, \beta_1 - u_1, \lambda)}{\sigma(u_3, u_1, \lambda)} \times \\ \times \exp\left(-u_3 \int_{\infty}^{(b, y(b))} \frac{(3x^3 + \lambda_4 x) dx}{-2y} - u_1 \int_{\infty}^{(b, y(b))} \frac{x^2 dx}{-2y}\right),$$

where (β_1, β_3) is the image of the point $(b, y(b))$ on the genus 2 curve (3) under the Abelian map

$$\beta_3 = \int_{\infty}^{(b, y(b))} \frac{dx}{-2y}, \quad \beta_1 = \int_{\infty}^{(b, y(b))} \frac{x dx}{-2y}.$$

The 1-forms $\frac{3x^3 + \lambda_4 x}{-2y} dx$ and $\frac{x^2}{-2y} dx$ are second kind differentials associated to the first kind differentials $\frac{1}{-2y} dx$ and $\frac{x}{-2y} dx$. The function $\Phi((u_3, u_1), (\beta_3, \beta_1))$ is a genus 2 analog of the elliptic Baker function (31).

Next we exploit the fact that $\Phi((u_3, u_1), (\beta_3, \beta_1))$ satisfies the equation

$$(47) \quad (\partial_{u_1 u_1} - 2\wp_{11})\Phi = b\Phi,$$

which is similar to a Schrödinger type equation

$$(48) \quad (\partial_{zz} - \mathcal{U}(z))\psi(z) = \mathcal{E}\psi(z)$$

where \mathcal{E} is a value of energy.

Corollary 2. Suppose $\lambda(\wp(\alpha), \gamma) \in \Lambda_1$ is defined by (35) with $\wp(\alpha) = A$. Then for all $U_3 \in \mathbb{C}$ the function

$$(49) \quad \psi(U_1) = \frac{\sigma(B_3 - U_3, B_1 - U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma))}{\sigma(U_3, U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma))} e^{U_1 \varpi},$$

where B_1 is an arbitrary complex number,

$$B_3 = \frac{1}{\wp'(\alpha)} \left(2\zeta(\alpha)B_1 + \log \frac{\sigma(\alpha - B_1)}{\sigma(\alpha + B_1)} \right), \\ \varpi = -\zeta(B_1) + \frac{1}{5}\wp(\alpha) \left(1 + \frac{18\zeta(\alpha)\wp(\alpha)}{5\wp'(\alpha)} \right) B_1 + \frac{9\wp(\alpha)^2}{25\wp'(\alpha)} \log \frac{\sigma(B_1 - \alpha)}{\sigma(B_1 + \alpha)}.$$

satisfies the Schrödinger equation (48) with the potential and energy

$$(50) \quad \mathcal{U}(U_1) = 2\mathcal{S}^2 - 2\wp(U_1) - 2\wp(\alpha), \quad \mathcal{E} = \wp(B_1),$$

where the notation (40b) is used.

Proof. Consider the equation (47) with respect to the variable $U_1 = u_1 - \frac{3}{5}\wp(\alpha)u_3$, and use \mathcal{P}_{11} from Corollary 1 instead of \wp_{11} . The equation acquires the form

$$(\partial_{U_1 U_1} - 2\mathcal{P}_{11})\Psi = b\Psi,$$

where Ψ is obtained from Φ by applying the substitution (39)

$$(51) \quad \Psi((U_3, U_1), (B_3, B_1)) = \frac{\sigma(B_3 - U_3, B_1 - U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma))}{\sigma(U_3, U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma))} \times \\ \times \exp\left(-U_3 \int_{\infty}^{(b, y(b))} dR_3 - U_1 \int_{\infty}^{(b, y(b))} dR_1\right),$$

where $B_3 = \beta_3$, $B_1 = \beta_1 - \frac{3}{5}\wp(\alpha)\beta_3$, the set of parameters $\lambda(\wp(\alpha), \gamma)$ is defined by (35) with $A = \wp(\alpha)$. Under the substitution (39) we get

$$B_3 = \int_{\infty}^{b + \frac{2}{5}\wp(\alpha)} \frac{dX}{-2(X - \wp(\alpha))Y(X)} = \int_0^{B_1} \frac{d\xi}{\wp(\xi) - \wp(\alpha)}.$$

The factor $\exp(-U_3 \int_{\infty}^{(b, y(b))} dR_3)$ is inessential so can be safely omitted. Next, we compute

$$dR_1 = \left(\frac{\frac{1}{5}\wp(\alpha)}{-2Y} + \frac{X}{-2Y} + \frac{\frac{9}{25}\wp(\alpha)^2}{-2Y(X - \wp(\alpha))} \right) dX$$

and obtain $\int_{\infty}^{(b, y(b))} dR_1 = \varpi$. Finally, using (39) we find $b = \wp(B_1) - \frac{2}{5}\wp(\alpha)$. \square

Remark 6. The function \mathcal{U} defined by (50) satisfies the KdV equation

$$4\partial_{U_3}\mathcal{U} = \partial_{U_1}^3\mathcal{U} - 6\mathcal{U}\partial_{U_1}\mathcal{U},$$

and is a stationary solution for higher equations of KdV hierarchy.

Suppose, the roots $\{e_j\}_{j=1}^5$, $\sum_j e_j = 0$, of polynomial $f(x, 0) = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}$, that is branch points of the curve (3), are real numbers, and $e_1 \geq e_2 \geq e_3 \geq e_4 \geq e_5$. Then the spectrum of operator in (47) is the union of three segments: $[e_5, e_4] \cup [e_3, e_2] \cup [e_1, \infty]$. When $\lambda \in \Lambda_1$ one of the segments, say $[e_5, e_4]$, contracts to produce a double point A . Under the conditions we can interpret the results of Corollary 2 in the following way.

Corollary 3. *Let (ω, ω') be periods of Weierstrass functions and assume $\text{Im } \omega = 0$, $\text{Re } \omega' = 0$. Then, provided $\wp(\alpha) \in \mathbb{R}$, formula (50) defines one parametric families, with parameter $\varphi \in [-\frac{1}{2}, \frac{1}{2}]$, of real-valued potentials in variable x on real line*

$$\begin{aligned} \mathcal{V}_1(x) &= \frac{1}{\omega^2} \mathcal{U}(\omega x), & \text{with } U_3 &= \frac{2\pi i}{\wp'(\alpha)} \varphi; \\ \mathcal{V}_2(x) &= \frac{1}{\omega^2} \mathcal{U}(\omega x + \tfrac{1}{2}\omega'), & \text{with } U_3 &= \frac{2\pi i}{\wp'(\alpha)} \varphi + \frac{1}{\wp'(\alpha)} (\zeta(\alpha)\omega' - \alpha\eta'). \end{aligned}$$

The operators $\partial_{xx} - \mathcal{V}_1(x)$ and $\partial_{xx} - \mathcal{V}_2(x)$ share a common spectrum

$$\{\wp(\alpha)\} \cup [\wp(\tfrac{1}{2}\omega'), \wp(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega')] \cup [\wp(\tfrac{1}{2}\omega), \infty].$$

Proof. Under the assumptions $\wp(z)$ is real when z runs from the origin along the boundary of rectangle with sides $\frac{1}{2}\omega$ and $\frac{1}{2}\omega'$. Further, both $(\wp(x), \wp'(x))$ and $(\wp(x + \frac{1}{2}\omega'), \wp'(x + \frac{1}{2}\omega'))$ are real for $x \in \mathbb{R}$. Let $\alpha \in (0, \frac{1}{2}\omega)$, the functions \mathcal{P} and \mathcal{S} defined by (40) are real-valued. At $\alpha \in (\frac{1}{2}\omega + \frac{1}{2}\omega', \frac{1}{2}\omega')$ value of $\wp'(\alpha)$ is real, and $\mathcal{P}/\mathcal{P}^* = 1$, as a result \mathcal{S} is real. At $\alpha \in (\frac{1}{2}\omega', 0) \cup (\frac{1}{2}\omega, \frac{1}{2}\omega + \frac{1}{2}\omega')$ values of $\wp'(\alpha)$ are imaginary, and $\mathcal{P}\mathcal{P}^* = 1$ so \mathcal{S} is imaginary. \square

Remark 7. The above potentials are unbounded except for $\mathcal{V}_2(x)$ with $\varphi \in (-\frac{1}{2}, \frac{1}{2})$ in three cases: (1) $\text{Re } \alpha = 0$, (2) $\text{Re } \alpha = \omega$, (3) $\text{Im } \alpha = 0$.

6.3. Rank 3 lattices. Consider the space \mathcal{C} of curves with a puncture at the common branch point at infinity. Choose the following basis of holomorphic differentials

$$(52) \quad h(x, y) = (1, x, -x^2, -(3x^3 + \lambda_4 x))^t \frac{dx}{-2y},$$

Denote by $\mathfrak{C} = (\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{b}_1)$ a basis of homology cycles such that $\mathfrak{a}_i \circ \mathfrak{b}_j = \delta_{ij}$, see Figure 1. Denote by Ω a matrix of integrals of $h(x, y)$ over \mathfrak{C} , that is $\Omega = \int_{\mathfrak{C}} h(x, y)$.

If $\lambda \in \Lambda_2$, then $\text{rank } \Omega = 4$ and Ω satisfies Legendre identity

$$(53) \quad \Omega^t J \Omega = 2\pi i J$$

for the symplectic matrix $J = \text{codiag}(1, 1, -1, -1)$. First two rows of Ω generate a rank 4 lattice in \mathbb{C}^2 , and thus define a two-dimensional complex torus as the quotient of \mathbb{C}^2 over the lattice. Meromorphic functions on the torus, that is *four-periodic functions on \mathbb{C}^2* , can be derived from sigma-function by taking logarithmic derivatives of order greater than 1. If $\lambda \in \Lambda_1 \cup \Lambda_0$, then $\text{rank } \Omega < 4$.

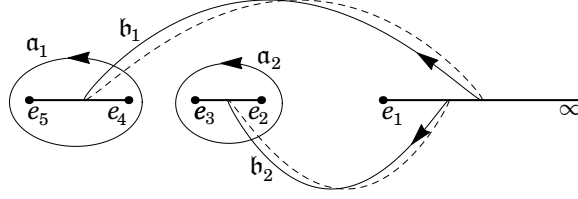


FIGURE 1. Sketch of branch points and basis homology cycles.

Introduce the notation

$$\mathcal{F}_k = \{\lambda \in \mathbb{C}^4 \mid \text{rank } \Omega = k\}, \quad k = 0, 1, 2, 3, 4.$$

Evidently, the space $\Lambda \cong \mathbb{C}^4$ is a disjoint union $\Lambda = \cup_{k=0}^4 \mathcal{F}_k$ (cf. Proposition 1), where $\mathcal{F}_4 = \Lambda_2$. Next,

Lemma 3. \mathcal{F}_3 is the set of simple roots of the discriminant $\Delta(\lambda)$ of (3)

$$\mathcal{F}_3 = \{\lambda \mid \Delta(\lambda) = 0, \partial_\lambda \Delta(\lambda) \neq 0\}.$$

Proof. Evidently, $\mathcal{F}_3 \subset \Lambda_1$. In the case of $\lambda \in \Lambda_1$ we use the transformations (39) to obtain elliptic parametrization $(x, y) = (\wp(\xi) - \frac{2}{5}\wp(\alpha), -\frac{1}{2}\wp'(\xi)(\wp(\xi) - \wp(\alpha)))$ with the uniformizing parameter $\xi \in \mathbb{C}$. Compute the integrals $I(x, y) = \int_\infty^{(x,y)} h(x, y)$ as functions in ξ

$$\begin{aligned} I_1(\xi) &= \frac{2\zeta(\alpha)}{\wp'(\alpha)}\xi + \frac{1}{\wp'(\alpha)} \log \frac{\sigma(\alpha - \xi)}{\sigma(\alpha + \xi)}, \\ (54) \quad I_2(\xi) &= \xi + \frac{3}{5}\wp(\alpha)I_1(\xi), \\ I_3(\xi) &= \zeta(\xi) - \frac{6}{25}\wp(\alpha)^2 I_1(\xi) - \frac{1}{5}\wp(\alpha)I_2(\xi), \\ I_4(\xi) &= -\frac{1}{2}\wp'(\xi) - \frac{3}{5}\wp(\alpha)(\gamma_4 + \frac{12}{25}\wp(\alpha)^2)I_1(\xi) - \frac{9}{25}\wp(\alpha)^2 I_2(\xi) - \frac{3}{5}\wp(\alpha)I_3(\xi). \end{aligned}$$

Now we calculate the periods. Let $\Omega = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 \\ H_1 & H_2 & H_3 & H_4 \end{pmatrix}$, where T_k and H_k are 2-dimensional vectors. By taking expansion of $I(\xi)$ near $\xi = \alpha$ we find that

$$\begin{pmatrix} T_1 \\ H_1 \end{pmatrix} = 2\pi i \text{Res}_{t=0} I(\alpha + t), \quad \begin{pmatrix} T_4 \\ H_4 \end{pmatrix} = \infty.$$

For this computations Figure 2 is instrumental.

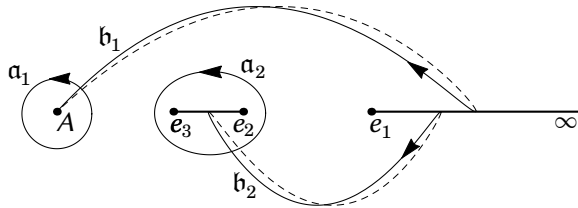


FIGURE 2. Sketch of branch points and basis homology cycles when two branch points contract.

On the other hand,

$$\begin{pmatrix} T_2 \\ H_2 \end{pmatrix} = I(\xi + \omega) - I(\xi), \quad \begin{pmatrix} T_3 \\ H_3 \end{pmatrix} = I(\xi + \omega') - I(\xi).$$

Explicitly, for finite periods we have

$$(55) \quad (T_1, T_2, T_3) = K_1 \begin{pmatrix} 0 & \omega & \omega' \\ -\frac{i\pi}{\alpha} & \eta & \eta' \end{pmatrix},$$

$$(56) \quad (H_1, H_2, H_3) = K_2(T_1, T_2, T_3) + K_3 \begin{pmatrix} 0 & \omega & \omega' \\ 0 & \eta & \eta' \end{pmatrix},$$

where

$$K_1 = \begin{pmatrix} \frac{2}{\wp'(\alpha)}\zeta(\alpha) & -\frac{2}{\wp'(\alpha)}\alpha \\ 1 + \frac{6}{5}\frac{\wp(\alpha)}{\wp'(\alpha)}\zeta(\alpha) & -\frac{6}{5}\frac{\wp(\alpha)}{\wp'(\alpha)}\alpha \end{pmatrix}, \quad K_2 = \begin{pmatrix} -\frac{9}{25}\wp(\alpha)^2 & 0 \\ 0 & -(\gamma_4 + \frac{12}{25}\wp(\alpha)^2) \end{pmatrix},$$

$$K_3 = \begin{pmatrix} -\frac{1}{5}\wp(\alpha) & 1 \\ \gamma_4 + \frac{6}{25}\wp(\alpha)^2 & -\frac{3}{5}\wp(\alpha) \end{pmatrix}.$$

and ω, ω' are periods of the Weierstrass function \wp , and $\eta = 2\zeta(\omega/2)$, $\eta' = 2\zeta(\omega'/2)$. Thus, when $\lambda \in \Lambda_1$ $\text{rank } \Omega = 3$ if and only if $\wp'(\alpha) \neq 0$.

To complete the proof it remains to notice that on Λ_1 the gradient of the discriminant (5) vanishes together with $\wp'(\alpha)$. Indeed,

$$(57) \quad \partial_\lambda \Delta(\lambda)|_{\Lambda_1} = \frac{1}{5}(4\gamma_4^3 + 27\gamma_6^2) (\wp'(\alpha))^6 \left(\left(\frac{3}{5}\wp(\alpha)\right)^3, \left(\frac{3}{5}\wp(\alpha)\right)^2, \frac{3}{5}\wp(\alpha), 1 \right),$$

here ∂_λ stands for $(\partial_{\lambda_4}, \partial_{\lambda_6}, \partial_{\lambda_8}, \partial_{\lambda_{10}})$. By definition $4\gamma_4^3 + 27\gamma_6^2$ does not vanish on Λ_1 . \square

Similarly, \mathcal{F}_2 is the set of double zeros of discriminant $\Delta(\lambda)$, and \mathcal{F}_1 is the set of triple zeros of discriminant $\Delta(\lambda)$:

$$\mathcal{F}_2 = \{\lambda \mid \partial_\lambda \Delta(\lambda) = 0, \partial_\lambda^2 \Delta(\lambda) \neq 0\},$$

$$\mathcal{F}_1 = \{\lambda \mid \partial_\lambda^2 \Delta(\lambda) = 0, \partial_\lambda^3 \Delta(\lambda) \neq 0\}.$$

Further, \mathcal{F}_0 is the set of 4-tuple zeros of discriminant $\Delta(\lambda)$ which is a single point $\lambda = 0$.

On the other hand, let $f(x) = x^5 + \sum_{k=0}^3 \lambda_{10-2k} x^k$. Divisor of zeros $(f)_0$ is a formal product $p_1^{d_1} p_2^{d_2} \dots p_5^{d_5}$, p_i are distinct points, integers d_i are non-negative and $\sum d_i = 5$ while $\sum d_i p_i = 0$. Assume $d_1 \geq d_2 \geq \dots \geq d_5$ and denote $\deg(f)_0 = (d_1, d_2, \dots)$, for nonzero d_i . Clearly, $\deg(f)_0$ takes values in partitions of number 5. Denote the dimension of corresponding subset of Λ by m , we have $m = \# \deg(f)_0 - 1$. In fact, m equals the dimension of a component of \mathcal{F}_m . Table 1 gives summary of all possible cases.

TABLE 1.

$\deg(f)_0$	(1, 1, 1, 1, 1)	(2, 1, 1, 1)	(3, 1, 1)	(2, 2, 1)	(3, 2)	(4, 1)	(5)
genus g	2	1	1	0	0	0	0
$\# \deg(f)_0 - 1$	4	3	2	2	1	1	0
$\text{rank } \Omega$	4	3	2	2	1	1	0

This completes description of the stratification of Λ by the rank of corresponding lattice.

Remark 8. Since \mathcal{F}_2 has nonempty intersections with both Λ_1 and Λ_0 , cf. Table 1, the two-periodic functions on the associated ‘torus’ can be of different nature: those that are essentially a combination of rational and elliptic functions, see Remark 4, and those that are combinations of exponential functions, see Theorem 2. The strata \mathcal{F}_1 and \mathcal{F}_0 are associated with exponential and rational functions respectively.

6.4. Three-periodic functions. On the stratum \mathcal{F}_3 in the place of identity (53) we have

$$\begin{pmatrix} T_1 & T_2 & T_3 \\ H_1 & H_2 & H_3 \end{pmatrix}^t J \begin{pmatrix} T_1 & T_2 & T_3 \\ H_1 & H_2 & H_3 \end{pmatrix} = 2\pi i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Corollary 4. For all $u = (u_3, u_1) \in \mathbb{C}^2$ sigma-function $\sigma(u; \lambda)$ obeys the periodicity property

$$\frac{\sigma(u \pm T_k; \lambda)}{\sigma(u; \lambda)} \Big|_{\lambda \in \mathcal{F}_3} = -\exp \left\{ \pm H_k^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (u \pm \frac{1}{2}T_k) \right\}, \quad k = 1, 2, 3.$$

Proof follows directly from the periodicity property of genus 2 sigma-function.

Remark 9. The function $\Phi(u, \beta)$ defined by (46) has Bloch property on $\mathcal{F}_4 = \Lambda_2$ and keeps the property when restricted to \mathcal{F}_3

$$\Phi(u + T_i, \beta) = \Phi(u, \beta) e^{M_i T_i}, \quad i = 1, 2, 3.$$

The “quasi-momenta” M_i are given by rather cumbersome expressions, which, however, can be readily deduced from (54), (55) and (56) in a condensed form. Let $\beta^t = (I_1(\alpha), I_2(\alpha))$ and $\rho = (I_4(\alpha), I_3(\alpha))$, where α is the image of double point A , that is $\wp(\alpha) = A$. Then we have

$$M_1 = \rho + \beta^t K_2, \quad M_2 = M_3 = \rho + \beta^t (K_2 + K_3 K_1^{-1}).$$

Now, return to discussing three-periodic functions. Over \mathcal{F}_3 any order greater than 1 logarithmic derivative of sigma-function is a three-periodic function.

Introduce the function

$$(58) \quad \mathcal{P}(u_3, u_1) = \frac{\sigma(\alpha + u_1 - \frac{3}{5}\wp(\alpha)u_3)}{\sigma(\alpha - u_1 + \frac{3}{5}\wp(\alpha)u_3)} e^{(\wp'(\alpha) + \frac{6}{5}\wp(\alpha)\zeta(\alpha))u_3 - 2\zeta(\alpha)u_1}$$

with $\wp'(\alpha) \neq 0$. It is straightforward to verify that T_1, T_2, T_3 are periods of the function $\mathcal{P}(u_3, u_1)$.

Corollary 5. Any meromorphic three-periodic function in two variables (u_3, u_1) with the periods T_1, T_2, T_3 is a rational function of $\mathcal{P}_{\text{basis}} = (\mathcal{P}(u_3, u_1), \wp(u_1 - \frac{3}{5}\wp(\alpha)u_3), \wp'(u_1 - \frac{3}{5}\wp(\alpha)u_3), \wp(\alpha), \wp'(\alpha))$.

Proof. Any genus 2 Abelian function, that is a meromorphic four-periodic function of (u_3, u_1) , has a unique representation as the rational function of $\wp_{\text{basis}} = (\wp_{11}, \wp_{13}, \wp_{111}, \wp_{113}, \wp_{1111}, \wp_{1113})$, in particular

$$(59) \quad \begin{aligned} \lambda_4 &= \frac{1}{2}\wp_{1111} - 3\wp_{11}^2 - 2\wp_{13}, \\ \lambda_6 &= \frac{1}{2}\wp_{1113} - \frac{1}{2}\wp_{1111}\wp_{11} + \frac{1}{4}\wp_{111}^2 + 2\wp_{11}^3 - 2\wp_{13}\wp_{11}, \\ \lambda_8 &= -\frac{1}{2}\wp_{1113}\wp_{11} - \frac{1}{2}\wp_{1111}\wp_{13} + \frac{1}{2}\wp_{113}\wp_{111} + \wp_{13}^2 + 4\wp_{11}^2\wp_{13}, \\ \lambda_{10} &= -\frac{1}{2}\wp_{1113}\wp_{13} + \frac{1}{4}\wp_{113}^2 + 2\wp_{13}^2\wp_{11}. \end{aligned}$$

The composition of $\Delta(\lambda)$, see (5), with (59) defines a polynomial $\Delta(\lambda(\wp_{\text{basis}}))$. When $\lambda \in \Lambda_1$, the polynomial $\Delta(\lambda(\wp_{\text{basis}}))$ should vanish, while $\delta(\gamma) = 4\gamma_4^3 + 27\gamma_6^2$ should be nonzero. Taking into account

$$\begin{aligned} \gamma_4 &= \frac{\wp'(U_1)^2 - 4\wp(U_1)^3 - (\wp'(\alpha)^2 - 4\wp(\alpha)^3)}{4(\wp(U_1) - \wp(\alpha))}, \\ \gamma_6 &= -\frac{\wp(\alpha)(\wp'(U_1)^2 - 4\wp(U_1)^3) - \wp(U_1)(\wp'(\alpha)^2 - 4\wp(\alpha)^3)}{4(\wp(U_1) - \wp(\alpha))}, \end{aligned}$$

where $U_1 = u_1 - \frac{3}{5}\wp(\alpha)u_3$, we see the condition $4\gamma_4^3 + 27\gamma_6^2 \neq 0$ turns into a condition on a polynomial in $\wp(U_1), \wp'(U_1), \wp(\alpha)$, and $\wp'(\alpha)$. By (38)–(41) we can express

$\wp_{11}, \wp_{13}, \wp_{111}, \wp_{113}$ as rational functions of $\mathcal{P}_{\text{basis}}$. Differentiating expressions for \wp_{111}, \wp_{113} with respect to u_1 we get the rational functions

$$\begin{aligned}\wp_{1111} &= 6\mathcal{S}^4 - 8(2\wp(U_1) + \wp(\alpha))\mathcal{S}^2 + 4\wp'(U_1)\mathcal{S} \\ &\quad + 4\wp(U_1)^2 + 10\wp(\alpha)\wp(U_1) + 4\wp(\alpha)^2 - \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{2(\wp(U_1) - \wp(\alpha))}, \\ \wp_{1113} &= -6(\wp(U_1) - \tfrac{2}{5}\wp(\alpha))\mathcal{S}^4 + 6\wp'(U_1)\mathcal{S}^3 \\ &\quad + \left(4\wp(U_1)^2 + \tfrac{38}{5}\wp(\alpha)\wp(U_1) + \tfrac{14}{5}\wp(\alpha)^2 - \frac{3(\wp'(U_1)^2 - \wp'(\alpha)^2)}{2(\wp(U_1) - \wp(\alpha))}\right)\mathcal{S}^2 \\ &\quad - \tfrac{2}{5}\wp'(U_1)\left(10\wp(U_1) + 11\wp(\alpha)\right)\mathcal{S} + 6\wp(\alpha)\left(\tfrac{2}{5}\wp(U_1)^2 + \wp(\alpha)\wp(U_1) + \tfrac{2}{5}\wp(\alpha)^2\right) \\ &\quad - \tfrac{4}{5}(\wp(U_1)^2 + \wp(\alpha)^2) + \frac{9\wp(U_1)(\wp'(U_1)^2 + \wp'(\alpha)^2)}{5(\wp(U_1) - \wp(\alpha))}.\end{aligned}$$

These rational expressions for \wp_{basis} substituted in $\Delta(\lambda(\wp_{\text{basis}}))$ make it vanish identically. Furthermore, we can re-express a rational function of \wp_{basis} as a rational function of $\mathcal{P}_{\text{basis}}$. \square

The parametrization of \wp_{basis} by rational functions of $\mathcal{P}_{\text{basis}}$ is analogous to the parametrization of λ in terms of a_2 and γ , cf. (13). In fact, the former parametrization is induced by the latter, which is clearly seen if we follow the connection between sigma-function σ and generators \wp_{basis} of the field of fiber-wise Abelian functions on the universal space of genus 2 Jacobi varieties.

Remark 10. Note that the function $f(z_1, z_2) = \mathcal{P}(z_1/\wp'(\alpha), cz_2 + \frac{3}{5}\wp(\alpha)z_1/\wp'(\alpha))$, where $c \neq 0$ is an arbitrary number, is a solution of the following system of functional equations

$$f(z_1, z_2)f(z_1, -z_2) = \exp(z_1), \quad f(z_1, z_2) = -f(-z_1, -z_2).$$

Proposition 2. *A field of three-periodic functions is a transcendental extension of the field of elliptic functions with transcendence degree 1.*

Proof follows from Corollary 5, the function $\mathcal{P}(u_3, u_1)$ serves as the transcendental element.

7. CONCLUDING REMARKS

For all values of parameters λ sigma-function $\sigma(u; \lambda)$ is essentially a function of the same nature, that is remains holomorphic and entire in all its arguments for singular curves as well. It possesses a periodicity property, in particular, for the case of actual genus 1 and $\partial_\lambda \Delta \neq 0$ it is given in Corollary 4, and the functions $-\partial_{u_i} \partial_{u_j} \log \sigma(u; \lambda)$ are three-periodic. The whole classification of degeneration strata is given in Table 1. ‘Degenerate’ expressions (24) and (33), at special values of parameters λ , are useful for solving generalized Jacobi inversion problem, and Schrödinger equation with periodic potential.

The technique we use above can be extended almost literally to higher genera hyperelliptic sigma-functions. Generalization to non-hyperelliptic sigma-functions is a challenging problem. Based on (24) and well-known formula for degenerate Weierstrass sigma-function [23], namely when $(g_2, g_3) \mapsto (12a^2, -8a^3)$

$$\sigma(u) \mapsto \tfrac{1}{2}\sqrt{3a}e^{-\frac{1}{2}au^2}(e^{\sqrt{3a}u} - e^{-\sqrt{3a}u}),$$

we conjecture that evaluation of a genus g hyperelliptic sigma-function at a stratum of parameters Λ_{g-1} , where genus of the underlying curve falls by 1, has similar structure

$$\sigma(u) \mapsto Ce^{-u^t \mathcal{Q}u} \left(\sigma(\mathcal{A} + u)e^{\mathcal{M}^t u} - \sigma(\mathcal{A} - u)e^{-\mathcal{M}^t u} \right).$$

Here sigma-function on the left hand side is in genus g , while sigma-function on the right hand side is in genus $g - 1$, scalar C , $g \times g$ matrix Q and vectors \mathcal{A} and \mathcal{M} are expressed with the help of first and second kind Abel integrals as functions of the coordinates of a double point and the parameters of genus $g - 1$ curve corresponding to a point in Λ_{g-1} . From this viewpoint sigma-function in genus 0 is a constant function, say, 1. We can regard the result of degeneration as an action of an operator \mathcal{T} , which is in essence an evaluation operator. Then properly tuned operators $\mathcal{T}(a)$ and $\mathcal{T}(b)$ associated with double points at a and b commute with respect to composition, which opens a possibility to study further degeneration of sigma-function in a more abstract setting.

For the generalized Jacobi inversion problem considered in Subsection 6.1 an alternative solution is known within the framework of the generalized Theta-function theory, which is developed by E. Previato [29], Yu. Fedorov [26], H. Braden and Yu. Fedorov [24]. A connection between the degenerate sigma-function (24) and the generalized Theta-function can be traced through the relation between sigma- and theta-functions in genus 1, see [23].

The subject of Subsection 6.2 may be viewed as the simplest examples of a potential of mixed solitonic and finite-gap nature. It is of considerable interest to explicitly construct potentials that possess arbitrary collection of points and segments in the place of spectra.

In general, lattices of odd ranks lead to generalized Jacobi varieties, see [25, 29]. The rank three lattice from Subsection 6.3 is an example of that. The corresponding generalized Jacobi variety is a union of a cylinder and a torus. At the same time, we conjecture that a field of $2g + 1$ -periodic functions can be effectively constructed as a transcendental extension of the field of hyperelliptic Abelian functions in genus g with help of a single transcendental element of a form similar to (58), namely

$$\mathcal{P}(u_{g+1}, u) = \frac{\sigma(\alpha + u)}{\sigma(\alpha - u)} \exp \{c(\alpha)u_{g+1} + d(\alpha)u\},$$

where σ denotes genus g sigma-function, $u, \alpha \in \mathbb{C}^g$, $u_{g+1} \in \mathbb{C}$, and $c(\alpha)$, $d(\alpha)$ are appropriate functions.

Our study of three-periodic functions of two complex variables will be extended in our future publications, in particular we plan to derive explicit form of addition law and to find special dynamical systems solvable by these functions.

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